XLIII. *A Letter to* James Weſt, *Eſq; Preſident of the Royal Society, containing the Inveſtigations of Twenty Caſes of Compound Intereſt, by* J. Robertſon, *Lib. R. S.*

S I R,

Read Dec. 13, 1770. IT is well known that the conſideration of Compound Intereſt is of great utility in the Computations reſpecting the values of Penſions, Annuities, Reverſions, and other affairs relating to Money concerns; and therefore, many Mathematicians have beſtowed ſome time on this uſeful ſubject, and have endeavoured to diſcover practical methods for ſolving the various caſes that might occur: And in theſe inquiries, the uſe of Logarithms has been found of ſingular ſervice in facilitating the operations.

The late William Jones, Eſq; F. R. S. among the variety of Mathematical matters to which he gave attention, conſidered the buſineſs of Compound Intereſt fully, and did, many years ago, cauſe to be engraved on a Copper-plate, more caſes in Intereſt than had been exhibited before that time: Several copies of impreſſions from that plate were diſtributed among his friends; to whom it appeared that he had treated this ſubject in a more extenſive manner than had been done by other Mathematicians.

The Theorems, or Rules, for the Caſes of Compound Intereſt, without their inveſtigations, were inſerted by Mr. Jones, in the quarto edition of Logarithms, publiſhed by Gardiner; and the Rules were alſo communicated to Mr. Dodſon, who publiſhed them, by Mr. Jones's leave, with examples to illuſtrate the uſe of his Antilogarithmic Tables: But the inveſtigations of theſe Theorems not having yet been made public, it is apprehended that Gentlemen curious in theſe ſpeculations would be pleaſed to ſee them: And ſhould the following Eſſay be approved of by you, who have been for many years acquainted with the Finances of this Nation, and the importance of a ready knowledge in computing the Intereſt on the National Loans, you will be pleaſed to communicate it to the Royal Society.

I am, SIR,

Your moſt humble ſervant,

John Robertſon.

In this ſubject five particulars are taken into conſideration.

1ſt, The Annuity, Rent, or Penſion.

2d, The Times that Annuity, Rent, or Penſion is to continue.

3d, The Rate of Intereſt uſed in the computation.

4th, The Amount of thoſe Rents, and their Intereſt, when they are forborn to be received any times after they are due.

5th, The preſent worth of thoſe Rents, ſome times before they are due; or, of a Sum to be received before it is due, Diſcount being allowed.

And the inveſtigations naturally fall under two heads.

Firſt, The Conſideration of Amounts.

Secondly, The Conſideration of Diſcounts.

Under the firſt head an Equation is to be obtained between the Annuity, Time, Rate and Amount, from the known proportion that ſubſiſts between ſums of money put to intereſt, during the ſame length of time, and the amounts of the principal and intereſt together.

Under the ſecond head another Equation is to be formed between the Annuity, Rate, Time and preſent Worth, from the known proportion that ſubſiſts between the ſums diſcounted, and their preſent worths, when done for the ſame time.

As theſe Equations involve quantities common to both of them, therefore other Equations may be thence deduced, containing all the five terms before ſpecified. And hence, any three of the five terms being given, the other two are to be found, which admits of 20 Caſes.

Some of theſe Caſes will produce Adfected Equations, where the index of the higheſt power of the unknown quantity will be the number of times the Rent is to continue, or to be paid: Therefore, the ſolution of thoſe Caſes will be given by a method of Approximation, as no better way has yet been diſcovered for the ſolution of Adfected Equations, in numbers, above the third or fourth degree.

In the following inveſtigations,

Let $a =$ Annuity, Rent, or Penſion.
$n =$ Number of times that intereſt is to be paid for the annuity, or ſum lent.
$r =$ Rate of intereſt of 1 £. for 1 time.
$m =$ Amount of the annuity, or ſum lent for n-times, at r intereſt.
$p =$ Principal ſum uſed, or preſent worth of a ſum before it is due.

Of Compound Intereſt.

Firſt, In Amounts,

Let $q = 1 + r =$ Amount of 1 £. for 1 time.
Now, $a =$ laſt year's amount.
And $1 : q :: a : aq =$ laſt but one year's amount.
$1 : q :: aq : aq^2 =$ laſt but two year's amount.
$1 : q :: aq^2 : aq^3 =$ laſt but 3.
And ſo on to $aq^{n-1} =$ firſt year's amount.

Therefore $a + aq + aq^2 + aq^3$ &c. $+ aq^{n-1} = m$.
But $a : aq :: m - aq^{n-1} : m - a$. Euc. 12. v.
Then $m - a = mq - aq^n$
Or $m - a = m + mr - aq^n$
Therefore $aq^n - a = mr$ Put $A = q^n$.
Then $\overline{A - 1} \times a = mr$

Therefore $A = \dfrac{mr + a}{a}$

Hence $a = \dfrac{mr}{A - 1}$

$$m = \frac{\overline{A - 1} \times a}{r}$$

$$r = \frac{\overline{A - 1} \times a}{m}.$$

Second,

Second, In Diſcounts.

Since $q : 1 :: a : \frac{a}{q} =$ 1ſt year's preſent worth.

$q : 1 :: \frac{a}{q} : \frac{a}{q^2} =$ 2d year's preſent worth.

$q : 1 :: \frac{a}{q^2} : \frac{a}{q^3} =$ 3d.

And ſo on to $\frac{a}{q^n} = n$th years preſent worth.

Therefore $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3} +$, &c. to $\frac{a}{q^n} = p$.

But $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}$. Euc. 12. v.

Therefore $\frac{pq^n - a}{q^n} = pq - a$.

Or $\frac{pq^n - a}{q^n} = p + pr - a$.

Therefore $aq^n - a = prq^n$.

Or $\overline{A - 1} \times a = prA$.

Therefore $A = \frac{a}{a - rp}$

Hence $\overline{A - 1} \times a = mr = prA$.

Therefore $A = \frac{mr + a}{a} = \frac{a}{a - rp} = \frac{m}{p} = \overline{1 + r}^n$.

Caſe I. Given a, m, p: Required r.

Since $A = \frac{m}{p}$.

Therefore $r = \frac{\overline{A - 1} \times a}{m}$.

Caſe II.

Caſe II. Given p, m, n: Required r.

Since $\overline{r+1}^{\,n} = \frac{m}{p}$.

Therefore $r = \overline{\frac{m}{p}}^{\,\frac{1}{n}} - 1$.

Caſe III. Given a, m, n: Required r.

Since $\overline{A-1} \times a = mr$.

Therefore $\left(\frac{A-1}{nr} =\right) \frac{m}{na} = \frac{\overline{1+r}^{\,n} - 1}{nr}$.

Now $\overline{1+r}^{\,n} = 1 + nr + n \times \frac{n-1}{2} r^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} r^3$, &c.

Therefore $\frac{m}{na} = 1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2$, &c.

And $\overline{\frac{m}{an}}^{\,\frac{2}{n-1}} = \overline{1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2}^{\,\frac{2}{n-1}}$; which, by the Binomial Theorem, will become $= 1 + r + \frac{n+1}{12} r^2$ (nearly).

Let $D = \left(\overline{\frac{m}{na}}^{\,\frac{2}{n-1}} =\right) 1 + r + \frac{n+1}{12} r^2$,

Then $r^2 + \frac{12}{n+1} r = \overline{D-1} \times \frac{12}{n+1}$, Let $2E = \frac{12}{n+1}$,

Then $r + E = \left(\sqrt{2 \times \overline{D-1+E} \times E} =\right) F$.

Therefore $r = F - E$.

In this Solution, 1ſt, find $D = \overline{\frac{m}{na}}^{\,\frac{2}{n-1}}$,

2d, find $E = \frac{6}{n+1}$,

3d, find $F = \sqrt{2 \times \overline{D-1+E} \times E}$,

4th, find $r = F - E$.

Caſe IV. Given a, p, n; Required r.

Since $\overline{A-1}\times a = Apr$.

Therefore $\frac{p}{na} = \frac{A-1}{Anr} = \frac{1-A^{-1}}{nr} = \frac{1-\overline{1+r}|^{-n}}{nr}$.

Now, $\overline{1+r}|^{-n} = 1 - nr + n\times\frac{n+1}{2}r^2 - n\times\frac{n+1}{2}\times\frac{n+2}{3}r^3 +$ &c,

Therefore $\frac{p}{na} = 1 - \frac{n+1}{2}r + \frac{n+1}{2}\times\frac{n+2}{3}r^2$, nearly.

Then $\overline{\frac{p}{na}}|^{-\frac{2}{n+1}} = \overline{1 - \frac{n+1}{2}r + \frac{n+1}{2}\times\frac{n+2}{3}r^2}|^{-\frac{2}{n+1}}$; which, by the Binomial Theorem, will become $= 1 + r - \frac{n-1}{12}rr$, nearly.

Now, $\overline{\frac{p}{na}}|^{-\frac{2}{n+1}} = \overline{\frac{na}{p}}|^{\frac{2}{n+1}}$.

Let $G = \left(\overline{\frac{na}{p}}|^{\frac{2}{n+1}} =\right) 1 + r - \frac{n-1}{12}r^2$.

Therefore $rr - \frac{12}{n-1}r = -\overline{G-1}\times\frac{12}{n-1}$.

Let $2H = \frac{12}{n-1}$.

Then $r - H = \left(\sqrt{\overline{H-2.\overline{G-1}}\times H} =\right) K$.

Therefore $r = H - K$.

Caſe V. Given a, n, r; Required m.

Since $\overline{A-1}\times a = mr$. Therefore $m = \frac{\overline{A-1}\times a}{r}$.

Caſe VI.

Caſe VI. Given p, n, r; Required m.

Since $A = \frac{m}{p}$. Therefore $m = pA$.

Caſe VII. Given a, p, r; Required m.

Since $\frac{m}{p} = \frac{a}{a - rp}$. Therefore $m = \frac{ap}{a - rp}$.

Caſe VIII. Given a, p, n; Required m.

Find $G = \overline{\frac{na}{p}}|^{\frac{2}{n+1}}$; $H = \frac{3}{\overline{\frac{1}{2}n - 1}}$; $K = \sqrt{\overline{H - 2\overline{G - 1}} \times H}$.

Now, $H - K = r$ by 4th. But $\overline{r + 1}|^{n} = \frac{m}{p}$.

Therefore $m = \overline{r + 1}|^{n} \times p$.

Caſe IX. Given a, n, r; Required p.

Since $\overline{A - 1} \times a = prA$. Therefore $p = \frac{\overline{A - 1} \times a}{Ar}$.

Caſe X. Given m, n, r; Required p.

Since $A = \frac{m}{p}$. Therefore $p = \frac{m}{A}$.

Caſe XI. Given a, m, r; Required p.

Since $\frac{mr + a}{a} = \frac{m}{p}$. Therefore $p = \frac{ma}{mr + a}$.

Caſe XII. Given a, m, n; Required p.

Find $D=\frac{\overline{m}^{\frac{2}{n-1}}}{na}$; $E=\frac{3}{\overline{\frac{1}{2}n+1}}$; $F=\sqrt{\overline{2\overline{D-1}+E}\times E}$

Now, $F-E=r$, by 3d. But $\overline{1+r}^{n}=\frac{m}{p}$.

Therefore $p=\frac{m}{\overline{1+r}^{n}}$.

Caſe XIII. Given p, n, r; Required a.

Since $\overline{A-1}\times a=prA$. Therefore $a=\frac{prA}{A-1}$.

Caſe XIV. Given p, m, n; Required a.

Since $A=\overline{r+1}^{n}=\frac{m}{p}$. Therefore $r+1=\overline{\frac{m}{p}}^{\frac{1}{n}}$.

Hence $A-1$, and $B-1$ ($=r$) are known.

Therefore $a=\frac{m\times\overline{B-1}}{A-1}$.

Caſe XV. Given m, n, r; Required a.

Since $\overline{A-1}\times a=mr$. Therefore $a=\frac{mr}{A-1}$.

Caſe XVI. Given p, m, r; Required a.

Since $A=\frac{m}{p}$, $A-1$ is given.

Therefore $a=\frac{mr}{A-1}$.

Caſe XVII. Given m, p, r; Required n.

Put L, for logarithm; and L̀, for arith. comp. of a log.

Since $\overline{1+r}^{\,n}=\frac{m}{p}$. Therefore $n=\frac{L,m+\grave{L},p}{L,\overline{r+1}}$.

Caſe XVIII. Given a, p, r; Required n.

Since $\overline{1+r}^{\,n}=\frac{a}{a-pr}$. Therefore $n=\frac{L,a-L,\overline{a-pr}}{L,\overline{r+1}}$.

Caſe XIX. Given a, p, m; Required n.

Since $A=\left(\overline{r+1}^{\,n}=\right)\frac{m}{p}$,

Then $\left(L,\overline{r+1}^{\,n}=\right) L,m-L,p=L,A$;

Hence A, and A — 1, are known.

Alſo $L,\overline{A-1}+L,a+\grave{L},m=L,\overline{B-1}$ (by Caſe 14th):

Hence $r=(B-1)$, and $r+1$, are known.

Therefore $n=\frac{L,A}{L,\overline{r+1}}$.

Caſe XX. Given a, m, r; Required n.

Since $(A=)\,\frac{mr+a}{a}=\overline{1+r}^{\,n}$.

Therefore $n=\frac{L,\overline{mr+a}-L,a}{L,\overline{1+r}}$.